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Review on Some Sequence Spaces of p-adic Numbers

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Abstract: In this paper, we make a literature review on p-adic sequences and we prove some new topological properties on the sequence spaces $\boldsymbol{w^{(p)}}$, $\boldsymbol{l_\infty}^{(p)}$ and $\boldsymbol{c^{(p)}}$ of p-adic numbers as the set of all sequences, the set of bounded sequences and the set of convergent sequences of p-adic numbers, respectively. We show that these sequence spaces are Banach spaces under some certain topological properties. Moreover we prove some inclusion relation between these sequence spaces. We construct the $\boldsymbol{\alpha-, \beta-}$ $\boldsymbol{and}$ $\boldsymbol{\gamma-}$duals of sequence spaces $\boldsymbol{w^{(p)}}$, $\boldsymbol{l_\infty}^{(p)}$ and $\boldsymbol{c^{(p)}}$ of p-adic numbers. We conclude the paper with characterizations of some significant matrix classes.

Keywords: Sequence Spaces, P-Adic Numbers, Banach Space, P-Adic Sequences, Matrix Transformations

1. Introduction

In the real case, we denote the space of all real valued sequences by ω. Each vector subspace of ω is called as a sequence space as well. The spaces of all bounded, convergent and null sequences are denoted by ℓ_∞, c and c_0, respectively. By ℓ_1, ℓ_p, cs, cs_0 and bs, we denote the spaces of all absolutely convergent, p-absolutely convergent, convergent, convergent to zero and bounded series, respectively; where $1 < p < \infty$ and p is a prime number.

A linear topological space λ is called a K-space if each of the map $\rho_i: \lambda \longrightarrow \mathbb{C}$ defined by $\mathrm{p_i(x) = x_i}$ is continuous for all $i \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0,1,2,3,\dots\}$. A K-space λ is called an FK-space if λ is a complete linear metric space. If an FK-space has a normable topology then it is called a BK-space. If λ is an FK-space, $\Phi \subset \lambda$ and (e^k) is a basis for λ then λ is said to have AK property, where (e^k) is the sequence whose only non-zero term is a 1 in k^{th} place for each $k \in \mathbb{N}$ and $\Phi = span\{e^k\}$. If Φ is dense in λ, then λ is called AD-space, thus AK implies AD.

Let λ and μ be two sequence spaces, and $A = (a_{nk})$ be an infinite matrix of real or complex numbers, where $\mathrm{n, k} \in \mathbb{N}$. For every sequence $X = (x_\mathrm{k}) \in \lambda$ the sequence $Ax = Ax = ((Ax)_\mathrm{n}) \in \mu$ is called A-transform of x, where

(1.1) $$(Ax)_{\mathrm{n}} = \sum_{k=0}^{\infty} a_{n_k} x_k .$$

Then, A defines a matrix mapping from λ to μ and we denote it by $A: \lambda \longrightarrow \mu$.

With $A \in (\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \longrightarrow \mu$ if and only if the series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax = ((Ax)_n)$ belongs to μ for all $x \in \lambda$. A sequence x is said to be A-summable to l and is called as the A-limit of x.

Let $\boldsymbol{\lambda}$ be a sequence space and A be an infinite matrix. The matrix domain λ_A of A in $\boldsymbol{\lambda}$ is defined by

$$\lambda_A = \{x = (x_k) \in \omega : Ax \in \lambda\}$$

which is a sequence space.

2. p-Adic Numbers

We begin with the definitions of p-adic numbers and p-adic integers with some topological properties.

Definition 2.1. (Katok, S. (2007)) In what follows p is a fixed prime number. The set $\mathbb{Q}_p$ is a completion of the rational numbers ($\mathbb{Q}$) with respect to the norm $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ given by

(2.1.1) $$|x|_p = \begin{cases} p^{-r} & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$$

where $x = p^r \frac{m}{n}$, $\forall x \in \mathbb{Q}$ and $m \in \mathbb{Z}$ and $n \in \mathbb{N}^+$ s.t. $(p, m) = (p, n) = 1$. The absolute value $|\cdot|_p$ is said to be non-Archimedean and the most important and useful property of this absolute value is satisfying the following inequality which is called "strong triangle inequality."

(2.1.2) $$|x + y|_p \leq \max\{|x|_p, |y|_p\}$$

i.e. if $|x|_p > |y|_p$ then $|x + y|_p = |x|_p$. This property is the most crucial property of non-Archimedean metric. Any p-adic number $x \in \mathbb{Q}_p$, where $x \neq 0$ is uniquely represented in the form

(2.1.3) $$x = p^r(x_0 + x_1 p^1 + x_2 p^2 + \cdots)$$

where $r \in \mathbb{Z}$ and x_i are integers, $0 \leq x_i < p,\ x_0 > 0, i = 0,1,2, \ldots$. This form is called the canonic form of $x \in \mathbb{Q}_p$ and $|x|_p = p^{-r}$.

Definition 2.2. (Sally, P. J. (1998)) Let $x \in \mathbb{Q}_p$ be a p-adic number. Then the following set is called the p-adic integers.

(2.2.1) $$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$$

For each $c \in \mathbb{Q}_p$, and $r > 0$, $B(c, r) = \{x \in \mathbb{Q}_p : |x - c|_p \leq r\}$ is called the p-adic ball centered at c with the radius of r.

3. The Sequence Spaces of p-adic Numbers

In this section, we may give the definitions of the sequence spaces of p-adic numbers and we contribute some topological properties under some certain conditions.

The sequence spaces of p-adic numbers were defined by (Katsaras, A. K. (1990)), as

$$w^{(p)} = \{x = (x_k) \text{ is any sequence in } \mathbb{Q}_p\},$$

$$l_\infty{}^{(p)} = \left\{x = (x_k) \in w^{(p)} : \sup_{k\in\mathbb{N}} |x_k|_p < \infty\right\},$$

$$c^{(p)} = \left\{x = (x_k) \in w^{(p)} : \exists l \in \mathbb{Q}_p, \ni \lim_{k\to\infty} |x_k - l|_p = 0\right\},$$

$$c_0^{(p)} = \left\{x = (x_k) \in w^{(p)} : \exists l \in \mathbb{Q}_p, \ni \lim_{k\to\infty} |x_k|_p = 0\right\},$$

as the set of all sequences of p-adic numbers, the set of all bounded sequences of p-adic numbers, the set of all convergent and null sequences of p-adic numbers, respectively.

Theorem 3.1. The set of all sequences of p-adic numbers $w^{(p)}$ is linear p-adic metric space with the metric

$$d_p(x,y) = \sum_k \frac{1}{p^k} \cdot \frac{|x_k - y_k|_p}{1 + |x_k - y_k|_p} \tag{3.1.1}$$

Proof. The linearity of the space can be seen easily since $w^{(p)}$ is closed under addition and scalar multiplication. So we skip the details. In order to show that the sequence space $w^{(p)}$ is a p-adic normed space with the norm defined by (3.1.1), we should show that the p-adic norm axioms are satisfied by the norm (3.1.1) for each element of the space $w^{(p)}$. Thus,

i) Let suppose that $d_p(x,y) = 0,\ for\ all\ x,y \in w^{(p)}$ which says that $|x_k - y_k|_p = 0,\ for\ every\ k \in \mathbb{N}_1$. Therefore we may say here that $(x_k - y_k) = 0,\ for\ every\ k \in \mathbb{N}_1$ says $x = y\ \ for\ every\ k \in \mathbb{N}_1$.
ii) One can be obtain from the properties of p-adic norm that $|x_k - y_k|_p = |y_k - x_k|_p$ holds $for\ every\ k \in \mathbb{N}_1$, so the second axiom of the p-adic norm $d_p(x,y) = d_p(y,x)$ holds $for\ all\ x,y \in w^{(p)}$.
iii) Let suppose that $x,y,z \in w^{(p)}$. Then the following inequality gives that

$$\begin{aligned}\frac{|x_k - y_k|_p}{1 + |x_k - y_k|_p} &= \frac{|x_k - z_k + z_k - y_k|_p}{1 + |x_k - z_k + z_k - y_k|_p} \\ &\le \frac{\max\{|x_k - z_k|_p, |z_k - y_k|_p\}}{1 + \max\{|x_k - z_k|_p, |z_k - y_k|_p\}} \\ &\le max\left\{\frac{|x_k - z_k|_p}{1 + |x_k - z_k|_p}, \frac{|z_k - y_k|_p}{1 + |z_k - y_k|_p}\right\}.\end{aligned}$$

When we take sum over $k \in \mathbb{N}_1$ from both side of the above inequality we have the following inequality

$$\sum_k \frac{1}{p^k}\frac{|x_k-y_k|_p}{1+|x_k-y_k|_p} \le max\left\{\sum_k \frac{1}{p^k}\frac{|x_k-z_k|_p}{1+|x_k-z_k|_p}, \sum_k \frac{1}{p^k}\frac{|z_k-y_k|_p}{1+|z_k-y_k|_p}\right\}$$

which says that $d_p(x,y) \le max\{d_p(x,z), d_p(z,y)\}$.

This result gives us that the sequence space $w^{(p)}$ is a p-adic metric space with the metric (3.1.1).

Theorem 3.2. The sequence spaces $c^{(p)}$, $c_0^{(p)}$ and $l_\infty{}^{(p)}$ of p-adic numbers is linear complete normed space with the norm

(2.2.1) $$\|x\|_\infty = \sup_k |x_k|_p$$

Proof. Since the linearity and the p-adic metric axioms can be shown easily, so we omit the repetition. We prove the theorem for only the sequence space $c^{(p)}$. Now, we should show that the sequence space $c^{(p)}$ is complete p-adic normed space with the norm defined by (2.2.1). In order to prove this, we must take a Cauchy sequence from the space $c^{(p)}$ and show that this Cauchy sequence is convergent to any number which is in $c^{(p)}$. Let us define a Cauchy sequence $x^j = (x_k^j)_{j\in\mathbb{N}} \in c^{(p)}$ for all k ∈ $\mathbb{N}_1$. Since every Cauchy sequence is convergent, there exists a p-adic number $x = (x_k) \in \mathbb{Q}_p$ and for every $\varepsilon > 0$ we have that

$$\left|x_k^j - x_k\right| < \varepsilon, \ \ for\ every\ j \in \mathbb{N}\ .$$

Therefore, we have the following inequality that

$$|x_k|_p = \left|x_k - x_k^j + x_k^j\right|_p$$

$$\le max\left\{\left|x_k - x_k^j\right|_p, \left|x_k^j\right|_p\right\}$$

When we pass to limit as $k \to \infty$, we have that $\lim_{k\to\infty}|x_k|_p$ exists which means that $x = (x_k) \in c^{(p)}$. This result concludes the proof.

Now we may give an example for the convergent sequences of p-adic numbers,

Example 3.3. Let us define a p-adic number $x = p^m\frac{a}{b},\ \ a \in Z, b \in Z^+\ and\ m \in Z$. The sequence of partial sum of the series $x_n = \sum_{k=0}^n x^k$ is convergent to $\frac{1}{1-x}$ if $m > 0$, but x_n is divergent if $m < 0$ with respect to the p-adic metric.

Theorem 3.5. The inclusion $c_0^{(p)} \subset c^{(p)}$ strictly holds.

Proof. To show the inclusion is strictly hold, we should define a sequence which belongs to $c^{(p)}$ but not in $c_0^{(p)}$. It can be easily shown with the above example 3.3. that the sequence $x_n = \sum_{k=0}^n x^k$ is convergent to $\frac{1}{1-x}$ if $m > 0$ which is not in the sequence space $c_0^{(p)}$ of p-adic numbers.

Corollary 3.6. (Natarajan, P. N. (2012)) The sequence spaces $c_0^{(p)}\ and\ c^{(p)}$ are closed subspaces of $l_\infty^{(p)}$.

4. $\alpha-, \beta-$ *and* $\gamma-$dual of the Sequence Spaces of p-adic Numbers

In this section, we define the duals of the sequence space with respect to the p-adic case with the definitions of series spaces of p-adic numbers. Now, we define the following sets

$$bs^{(p)} = \left\{ x = (x_k) \in w^{(p)} : \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n} x_k \right|_p < \infty \right\},$$

$$cs^{(p)} = \left\{ x = (x_k) \in w^{(p)} : \exists l \in \mathbb{Q}_p, \lim_{n\to\infty} \left| \sum_{k=0}^{n} x_k - l \right|_p = 0 \right\},$$

$$l_1^{(p)} = \left\{ x = (x_k) \in w^{(p)} : \left| \sum_{k} x_k \right|_p < \infty \right\},$$

as the series space of p-adic numbers whose sequences of the partial sum is bounded, the series space of p-adic numbers whose sequences of the partial sum is convergent, and the space of absolutely summable series of p-adic numbers, respectively. By (Gouvêa, F. Q. (1997)) we may say that these spaces are complete metrizable topological vector spaces of p-adic numbers with respect to the p-adic norm defined by (2.2.1).

Remark: It can be obtained from here that all the series $\sum_k x_k$ converges if and only if $x_k \to 0 \ as\ k \to \infty$. Therefore, $c_0^{(p)}$ coincides with the space of all convergent series $cs^{(p)}$.

Let μ be any sequence space of p-adic numbers. Then we may define the duals of any sequence space in p-adic case.

$$\{\mu^{(p)}\}^{\alpha} = \{a = (a_k) \in w^{(p)} : ax = (a_k x_k) \in l_1^{(p)}, \text{ for all } x = (x_k) \in \mu\}$$

$$\{\mu^{(p)}\}^{\beta} = \{a = (a_k) \in w^{(p)} : ax = (a_k x_k) \in cs^{(p)}, \text{ for all } x = (x_k) \in \mu\}$$

$$\{\mu^{(p)}\}^{\gamma} = \{a = (a_k) \in w^{(p)} : ax = (a_k x_k) \in bs^{(p)}, \text{ for all } x = (x_k) \in \mu\}$$

5. Matrix Transformations on the Sequence Space of p-adic Numbers

In this present chapter, we give the needed definitions and theorems for the characterizations of the matrix classes related with the sequence spaces $c^{(p)}$, $c_0^{(p)}$ and $l_\infty{}^{(p)}$ and we review the important metric classes and their characterizations.

Definition 5.1. (Andree, R. V., & Petersen, G. M. (1956)**)** A sequence $s = (s_n)$ obtained from the infinite matrix $A = (a_{mn})$ and the sequence $x = (x_n)$ using the relation $s_n = \sum_{k=0}^{n} a_{mk}\, x_k$ is called $A = (a_{mn})$ transform of $x = (x_n)$. If $s = (s_n)$ converges to T, the matrix $A = (a_{mn})$ is said to be sum of the sequence $x = (x_n)$ to the sum T.

Definition 5.2. (Andree, R. V., & Petersen, G. M. (1956)**)** The method of summation defined by the matrix $A = (a_{mn})$ is called regular in the p-adic field $\mathbb{Q}_p$ if every convergent sequence $x = (x_n)$ is equal to its transform $s = (s_n)$ in $\mathbb{Q}_p$. The sequences $x = (x_n)$ and $s = (s_n)$ are equal if

$\lim_{n\to\infty}|x_n - s_n|_p = 0$. Clearly if $x = (x_n)$ is p-convergent, then $s = (s_n)$ is p-convergent. The inverse need not hold.

Theorem 5.3. (Andree, R. V., & Petersen, G. M. (1956)**)** The matrix $A = (a_{mn})$ is called p-regular if and only if the following conditions hold.

(5.3.1) $$\lim_{m\to\infty} |a_{mn}|_p = 0$$

(5.3.2) $$\lim_{n\to\infty} \left|\sum_{k=0}^{n} a_{mk}\, x_k - 1\right|_p = 0$$

(5.3.3) $$|a_{mn}|_p \leq M$$

The method $s_n = \sum_{k=0}^{n} a_{mk}\, x_k$ is also called p-regular.

Theorem 5.4. Let $A = (a_{nk})$ be an infinite matrix. Then $A \in (c^{(p)}: l_1^{(p)})$ if and only if

(5.4.1) $$\sup_{K\in F} \sum_n \left|\sum\nolimits_{k\in K} a_{nk}\right|_p < \infty$$

where F is finite subset of $\mathbb{N}$.

Theorem 5.5. Let $A = (a_{nk})$ be an infinite matrix. Then $A \in (c^{(p)}: l_\infty^{(p)})$ if and only if

(5.5.1) $$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}|_p < \infty$$

Theorem 5.6. (Monna, A. F. (1970)) Let $A = (a_{nk})$ be an infinite matrix. Then $A \in (c^{(p)}: c^{(p)})$ if and only if

(5.6.1) $$\sup_{n,k\in\mathbb{N}} |a_{nk}|_p < \infty$$

(5.6.2) $$\exists a_k \in \mathbb{Q}_p \; such\ that \; \lim_{n\to\infty} |a_{nk} - a_k|_p = 0, \;\; for\ all\ k \in \mathbb{N}.$$

(5.6.3) $$\exists a \in \mathbb{Q}_p \; such\ that \; \lim_{n\to\infty} \left|\sum_k a_{nk} - a\right|_p = 0.$$

Theorem 5.7. (Natarajan, P. N. (2012)) Let $A = (a_{nk})$ be an infinite matrix. Then $A \in (l_\infty^{(p)}: c_0^{(p)})$ if and only if

(5.7.1) $$\lim_{k\to\infty} |a_{nk}|_p = 0\,, for\ n = 0,1,2,3, \ldots$$

(5.7.2) $$\lim_{n\to\infty} \sup_{k\geq 0} |a_{nk}|_p = 0$$

Theorem 5.8. (Natarajan, P. N. (2012)) Let $A = (a_{nk})$ be an infinite matrix. Then $A \in (l_\infty^{(p)}: c^{(p)})$ if and only if (5.7.1) holds and

(5.8.1) $$\lim_{n\to\infty}\sup_{k\geq 0}|a_{n+1,k}-a_{nk}|_p=0$$

6. Conclusion

The concept of p-adic numbers has been studied by many of the mathematicians as p-adic algebra and as p-adic analysis(see (Parent, D. P. (1984)),(Khrennikov, A. (1997)), (Koblitz, N. (1977))). Most of the important topological and algebraic properties were discovered and introduced. Nowadays, there are significant applications of p-adic numbers and p-adic analysis in pure mathematics, mathematical physics, applied statistics and computer sciences. In this paper, we tried to build up some new topological properties of sequence spaces of p-adic numbers. Moreover, we introduced the $\alpha-, \beta - and\ \gamma -$dual of any sequence space of p-adic numbers. The characterization of some significant matrix classes and matrix transformations on the sequence spaces of p-adic numbers are mentioned. This paper is significant review for the following research papers. The matrix transform over the sequence spaces of p-adic numbers are still open problems.

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